propagates to the right, whereas an expansion wave starts at $x = 1$ and propagates to the left The pattern has a period $\Delta t = 2$, or the time required for an acoustic wave to travel to the opposite end of the tank and return

Computation for $t = 2$ to 4 of Eq (16) shows that it is valid for all $t > 0$, and the restriction $t > x + 1$ may be removed if $g(z)$ is extended to negative z by its periodicity Note that the pressure in the center of the tank is not influenced by the acceleration

For the incompressible fluid, we see from Eq (7) and the boundary conditions that $u \equiv 0$, and the linearized momentum equation after integration becomes $(p - p_0) = -\rho_0 a_0 x$ Thus the acceleration acts upon the fluid as a time varying gravitational field We easily recognize that in Eq (16) the solution is separated into 3 parts: the first represents the hydrostatic pressure term, the other two terms represent right and left propagating waves

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Approximate Solution to Flux Concentration by Hydromagnetic Flow

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Nomenclature

B = magnetic flux density
E = electric field intensity
Ei = exponential integral $\left(\int_{-\infty}^{-x} t^{-1}e^t dt\right)$ tabulated in Ref 6
f = function defined by Eq (10)
J = current density
L = characteristic length equal to outer radius of annulus
p = pressure
R_m = magnetic Reynolds number defined by Eq (15)
r = radius
u = velocity component in radial direction
V = velocity vector
z = axial direction
μ_0 = magnetic permeability ($4\pi \times 10^{-7}$ henry/m)
ν = kinematic viscosity
ξ = dimensionless radius
ρ = fluid density
σ = fluid electrical conductivity
φ = azimuthal direction

Introduction

FLUX concentration by the interaction of a flowing conductor with an applied magnetic field has been investigated previously[1–3] Results were obtained either by numerical methods or by rather complicated mathematical expressions which were limited to low magnetic Reynolds numbers This note presents a simple approximate solution which is valid for all magnetic Reynolds numbers and is in good agreement with the previously presented numerical results

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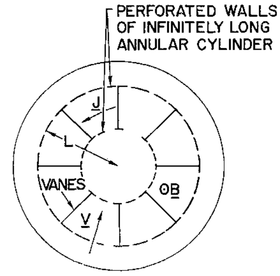

Fig 1 Hydromagnet geometry

The assumed geometry of the hydromagnet is shown in Fig 1 A conducting fluid flows radially inward between vanes in an infinitely long annular cylinder An axially applied magnetic field induces an azimuthal electric current, giving rise to an induced magnetic field in the same direction as the applied field The ultimate operating characteristics of the device depend on the properties of the conducting fluid, the magnetic Reynolds number, and the ratio of the internal and external radii of the annulus

Analysis

To analyze the flow, a cylindrical coordinate system with the z axis coincident with the hydromagnet is assumed The conducting fluid is assumed to be incompressible, and the fluid flow between the vanes is purely radial It also is assumed that the magnetic field has a z component only

Under these restrictions, the momentum equations in the r and φ directions are, respectively,[3]

$$u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{1}{\rho}(\mathbf{J}\times\mathbf{B})_r + \nu\left(\nabla^2 u - \frac{u}{r^2}\right) \quad (1)$$

$$0 = -\frac{1}{r\rho}\frac{\partial p}{\partial \varphi} + \frac{1}{\rho}(\mathbf{J}\times\mathbf{B})_\varphi + 2\frac{\nu}{r^2}\frac{\partial u}{\partial \varphi} \quad (2)$$

For steady-state operation, Maxwell's equations in rationalized mks units are

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad (3)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (4)$$

$$\nabla \times \mathbf{E} = 0 \quad (5)$$

Finally, Ohm's law is given by

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{V}\times\mathbf{B}) \quad (6)$$

The electromagnetic body force per unit fluid volume can be written, after some vector manipulation, as

$$\mathbf{J}\times\mathbf{B} = -\frac{1}{2\mu_0}\nabla B^2 + \frac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} \quad (7)$$

In view of the assumption that the magnetic field has a z component only, the last term in Eq (7) can be shown to vanish Then, use of the resulting expression in Eqs (1) and (2) yields

$$u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial}{\partial r}\left(p + \frac{B^2}{2\mu_0}\right) + \nu\left(\nabla^2 u - \frac{u}{r^2}\right) \quad (8)$$

$$0 = -\frac{1}{\rho r}\frac{\partial}{\partial r}\left(p + \frac{B^2}{2\mu_0}\right) + \frac{2\nu}{r^2}\frac{\partial u}{\partial \varphi} \quad (9)$$

The equation of continuity for purely radial, incompressible flow reduces to

$$u = f(\varphi)/r \quad (10)$$

Substitution of Eq (10) into (8) and (9) gives

$$-\frac{f^2}{r^3} = -\frac{1}{\rho}\frac{\partial}{\partial r}\left(p + \frac{B^2}{2\mu_0}\right) + \frac{\nu}{r^3}\frac{\partial^2 f}{\partial \varphi^2} \quad (11)$$

$$0 = -\frac{1}{\rho r}\frac{\partial}{\partial\varphi}\left(p + \frac{B^2}{2\mu_0}\right) + 2\frac{\nu}{r^3}\frac{\partial f}{\partial\varphi} \qquad (12)$$

However, as shown in Refs 1 and 3, f is essentially independent of φ except very near the radial vane surface Hence, as a first approximation, one may assume that f is independent of φ This is equivalent to assuming that all viscous effects are confined to the boundary layer near the vane surface

With f a constant, the velocity is independent of φ, and it can be shown that the magnetic field is also independent of φ Elimination of **J** from Eqs (3) and (6), taking the curl of the resulting expression, and utilizing Eq (5) yields

$$\nabla^2\mathbf{B} + \mu_0\sigma\nabla \times (\mathbf{V} \times \mathbf{B}) = 0 \qquad (13)$$

By use of the previously stated assumptions, Eq (13) reduces to

$$\frac{d^2B}{dr^2} + \frac{1}{r}\frac{dB}{dr} - \frac{\mu_0\sigma f}{r}\frac{dB}{dr} = 0 \qquad (14)$$

Reference 4 shows that $f = -r|u_{\max}|$ Let $r = \xi L$, and define a magnetic Reynolds number by[1 3]

$$R_m = \mu_0\sigma|u_{\max}|L \qquad (15)$$

Then Eq (14) becomes

$$\frac{d^2B}{d\xi^2} + \left(\frac{1}{\xi} + R_m\right)\frac{dB}{d\xi} = 0 \qquad (16)$$

Integrating once with the boundary condition[1] that at $\xi = 1\ 0$

$$dB/d\xi = -R_mB_0$$

where B_0 is the initially applied magnetic field, one obtains

$$dB/B_0 = -(R_m/\xi)e^{Rm(1-\xi)}\,d\xi \qquad (17)$$

The solution of this equation,[5] with the boundary condition that $B = B_0$ at $\xi = 1\ 0$[1], is

$$B/B_0 = 1 + [Ei(-R_m) - Ei(-R_m\xi)]R_me^{R_m} \qquad (18)$$

Equation (18), which is a much simpler expression than that given in Refs 1 and 3, is valid for all magnetic Reynolds numbers and involves no more work than looking up previously tabulated functions (e g , Ref 6)

Results and Discussion

The magnetic field ratio for several magnetic Reynolds numbers is shown in Fig 2 as a function of the ratio of the inner to outer radius of the annulus The results of Refs 2 and 3 also have been included for comparison The R_m = 0 8, 1 0, and 2 0 curves were computed by numerical methods,[2] while the R_m = 0 2 and 0 6 curves were obtained by an analytic solution[3] involving a series expansion in R_m Note that the simple approximate solution given by Eq (18) is in good agreement with the results of Ref 2, as well as those of Ref 3 for small R_m The results of Ref 3 diverge considerably from Eq (18) with increasing R_m However, as noted previously, Eq (18) is valid for all R_m, whereas the results of Ref 3 are not Thus, Fig 2 shows that the solution given by Ref 3 is apparently limited to very small values of magnetic Reynolds number

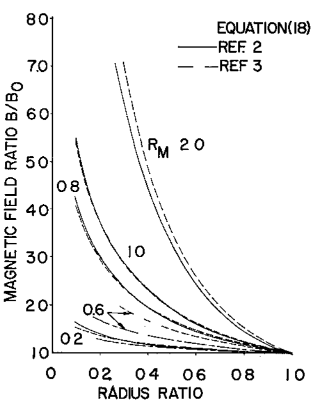

Fig 2 **Magnetic field ratio**

It is seen in Fig 2 that quite substantial increases in magnetic field can be obtained, even with a small-sized device and with rather low fluid velocities Assuming that the conducting fluid is liquid sodium, that $u_{\max}$ = 0 3 m/sec, and that L = 0 3 m, R_m is approximately 1 1 The corresponding conventional Reynolds number is approximately 4 4 × 10[5], indicating that the viscous effects are indeed confined to a thin layer near the vane surface The limits on generated fields, as well as practical considerations such as viscous and ohmic dissipation, are discussed in Ref 3

After this work was nearly completed, it was discovered that the same inviscid approximation as that above had been made in Ref 7, leading to an equation identical to Eq (16) In Ref 7, however, a solution was obtained using the boundary condition that *all* magnetic flux lines were limited to the region inside the outer radius of the annulus For large R_m, this would lead to prohibitively large current densities in a narrow region near the inner radius The boundary condition used in Refs 1 and 3, as well as in the present work, is that the magnetic field at the outer radius is equal to the applied field This implies that the magnetic flux lines return in a region exterior to the hydromagnet annulus, thereby reducing the possibility of large current densities near the inner radius It should be pointed out that the latter boundary condition also was used in Ref 7 to obtain a solution (identical to Refs 1 and 3) for the case including viscous effects

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Orbital Transfer with Minimum Fuel

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A NOTE in this Journal[1] discussed the problem of scheduling the direction p of constant momentum thrust of a rocket, which loses mass at a constant rate, so that it trans-

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